

CONTROLLABILITY OF COMPLEX NETWORKS USING PERTURBATION THEORY OF EXTREME SINGULAR VALUES

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ABSTRACT. Pinning control on complex dynamical networks has emerged as a very important topic in recent trends of control theory due to the extensive study of collective coupled behaviors and their role in physics, engineering and biology. In practice, real-world networks consists of a large number of vertices and one may only be able to perform a control on a fraction of them only. Controllability of such systems has been addressed in [6], where it was reformulated as a global asymptotic stability problem. The goal of this short note is to refine the analysis proposed in [6] using recent results in singular value perturbation theory.

1. INTRODUCTION

In recent years, extensive efforts have been devoted to the control of complex dynamical networks. One major issue is that real world networks usually consist of a very large number of nodes and links which makes it impossible to apply control actions to all nodes.

Pinning control is a new way to address this problem by placing local feedback injections on a small fraction of the nodes.

Controllability of such systems has been addressed in [6], where it was reformulated as a global asymptotic stability problem. The goal of this short note is to refine the analysis proposed in [6] using recent results in singular value perturbation theory.

1.1. The model. One considers a set of N n -dimensional oscillators governed by a system of nonlinear differential equations. Moreover, we assume that each oscillator is coupled with a restricted set of other oscillators. This coupling relationship can be efficiently described using a graph where the vertices are indexed by the oscillators and there is an edge between two oscillators if they are coupled. The overall dynamical system is given by the following set of differential equations

$$(1.1) \quad x_i'(t) = f(x_i(t)) - \sigma B \sum_{j=1}^N l_{ij} x_j(t) + u_i(t), \quad t \geq t_0,$$

$i = 1, \dots, N$, where $x_i(t) \in \mathbb{R}^n$ is the state of the i^{th} oscillator, $\sigma > 0$, $B \in \mathbb{R}^{n \times n}$, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ describes the dynamics of each oscillator, $L = (l_{ij})_{i,j=1,\dots,N}$ is the graph Laplacian of the underlying graph, and $u_i(t)$, $i = 1, \dots, N$ are the controls. For the system to be well defined, we have to specify some initial conditions $x_i(t_0) = x_{i0}$ for $i = 1, \dots, N$.

1.2. The control problem. Assume that we have a reference trajectory $s(t)$, $t \geq t_0$ satisfying the differential equation

$$s'(t) = f(s(t)).$$

Our goal is to control the system using a limited number of nodes. The selected nodes are called the "pinned nodes". For this purpose, we use a linear feedback law of the form

$$u_i(t) = p_i K e_i(t),$$

where $e_i(t) = s(t) - x_i(t)$, K is a feedback gain matrix, and where

$$p_i = \begin{cases} 1 & \text{if node } i \text{ is pinned} \\ 0 & \text{otherwise.} \end{cases}$$

Let P denote the diagonal matrix with diagonal (p_1, \dots, p_N) .

1.3. Controllability. In [6], the authors propose a definition for (global pinning-) controllability (based on Lyapunov stability criteria):

Definition 1.1. *We say that the system (1.1) is controllable if the error dynamical system $e := (e_i(t))_{1 \leq i \leq N}$ is Lyapunov stable around the origin, i.e. there exists a positive definite function V such that $\frac{d}{dt}V(e(t)) < 0$ when $e(0) \neq 0$.*

The following result, [6, Corollary 5], provides a sufficient condition for a system to be controllable:

Proposition 1.2 ([6]). *Assume that f is such that there exists a bounded matrix $F_{\xi, \tilde{\xi}}$, whose coefficients depend on ξ and $\tilde{\xi}$, which satisfies*

$$(1.2) \quad F_{\xi, \tilde{\xi}}(\xi - \tilde{\xi}) = f(\xi) - f(\tilde{\xi}), \quad \xi, \tilde{\xi} \in \mathbb{R}^n.$$

Let $Q \in \mathbb{R}^{n \times n}$ be a positive definite matrix such that

$$\begin{aligned} QK + K^t Q^t &= \kappa (QB + B^t Q^t) \\ (QB + B^t Q^t) &\succeq 0 \end{aligned}$$

and

$$(1.3) \quad \frac{1}{2} \lambda_N(\sigma L + \kappa P) \lambda_n(QB + B^t Q^t) > \sup_{\xi, \tilde{\xi}} \|F_{\xi, \tilde{\xi}}\| \|Q\|.$$

Then the system is controllable.

Many systems of interest satisfy the constraint specified by (1.2); see [?]. This proposition is very useful for node selection via the matrix P . Indeed, assume that Q is selected, then one may try to maximise $\lambda_N(\sigma L + \kappa P)$ as a function of P , under the constraint that no more than r nodes can be pinned. This is a combinatorial problem that can be relaxed using semi-definite programming or various heuristics [3].

1.4. Goal of the paper. Our goal in the present note is to propose an easy controllability condition refining [6, Corollary 7], based on the algebraic connectivity of the graph, the number of pinned nodes, the coupling strength and the feedback gain. Our approach is based on perturbation theory of the extreme singular values of a matrix after appending a column. The basic results of this theory are given in the appendix.

2. MAIN RESULT

2.1. Notations. The Kronecker symbol is denoted by $\delta_{i,j}$, i.e. $\delta_{i,j} = 1$ if $i = j$ and is equal to zero otherwise. We denote by $\|x\|_2$ the euclidian norm of a vector x and by $\|A\|$ the associated operator norm (spectral norm) of a matrix A .

For any symmetric matrix $B \in \mathbb{R}^{d \times d}$ we will denote its eigenvalues by $\lambda_1(B) \geq \dots \geq \lambda_d(B)$. The largest eigenvalue will sometimes also be denoted by $\lambda_{\max}(B)$ and the smallest by $\lambda_{\min}(B)$. The smallest nonzero eigenvalue of a positive semi-definite matrix B will be denoted by $\lambda_{\min>0}(B)$.

2.2. A simple criterion for controllability. Our main result is the following theorem.

Theorem 2.1. *Let $Q \in \mathbb{R}^{n \times n}$ be a positive definite symmetric matrix that satisfies*

$$\begin{aligned} QK + K^t Q^t &= \kappa (QB + B^t Q^t) \\ (QB + B^t Q^t) &\succeq 0, \end{aligned}$$

and assume that

$$(2.4) \quad \|F_{\xi, \tilde{\xi}}\| < \frac{\sigma \lambda_{\min>0}(L) \lambda_{\min}(QB + B^t Q^t)}{2 \|Q\|}.$$

If κ satisfies

$$\kappa \geq \frac{\sum_{i=1}^r \deg_i}{\sigma \lambda_{\min>0}(L) - \frac{2 \|F_{\xi, \tilde{\xi}}\| \|Q\|}{\lambda_{\min}(QB + B^t Q^t)}} + \sigma \lambda_{\min>0}(L),$$

then the system is controllable.

Proof. We follow the same steps as for the proof of Corollary 7 in [6]. We assume without loss of generality that the first r nodes are the pinned nodes. We may write P as

$$P = \sum_{i=1}^r e_i e_i^t,$$

where e_i is the i^{th} member of the canonical basis of \mathbb{R}^N , i.e. $e_i(j) = \delta_{i,j}$. We will try to compare $\lambda_N(\sigma L + \kappa P)$ with $\lambda_N(\sigma L)$ and use Proposition 1.2 to obtain a sufficient condition for controllability based on L , i.e. the topology of the network. For this purpose, let us recall that L can be written as

$$L = \mathcal{I} \cdot \mathcal{I}^t,$$

where \mathcal{I} is the incidence matrix of any directed graph obtained from the system's graph by assigning an arbitrary sign to the edges [1]. Of course L will not depend on the chosen assignment. Using this factorization of L , we obtain that

$$\sigma L + \kappa \sum_{i=1}^r e_i e_i^t = [\sqrt{\kappa} e_r, \dots, \sqrt{\kappa} e_1, \sqrt{\sigma} \mathcal{I}] [\sqrt{\kappa} e_r, \dots, \sqrt{\kappa} e_1, \sqrt{\sigma} \mathcal{I}]^t.$$

Moreover, $\lambda_{\min>0}(\sigma L + \kappa P)$ can be expressed easily as the smallest nonzero eigenvalue of the r^{th} term of a sequence of matrices with shape (A.8) for which we can use Theorem A.2 iteratively. Indeed, we have

$$\lambda_{\min>0}(\sigma L + \kappa e_1) = \lambda_{\min>0} \left([\sqrt{\kappa} e_1, \sqrt{\sigma} \mathcal{I}]^t [\sqrt{\kappa} e_1, \sqrt{\sigma} \mathcal{I}] \right).$$

Let us denote by x the vector $\sqrt{\kappa} e_1$ and by X the matrix $[\sqrt{\sigma}\mathcal{I}]$. Then, we have that

$$[\sqrt{\kappa} e_1, \sqrt{\sigma}\mathcal{I}]^t [\sqrt{\kappa} e_1, \sqrt{\sigma}\mathcal{I}] = \begin{bmatrix} x^t x & x^t X \\ X^t x & X^t X \end{bmatrix}.$$

Therefore, Theorem A.2 gives

$$\lambda_{\min>0}(\sigma L + \kappa e_1 e_1^t) \geq \sigma \lambda_{\min>0}(L) - \frac{\deg_1}{(\kappa - \sigma \lambda_{\min>0}(L))},$$

where \deg_1 is the degree of node number 1.

Let us now consider $\lambda_{\min>0}(\sigma L + \kappa e_1 + \delta_2 e_2)$. We have that

$$\lambda_{\min>0}(\sigma L + \kappa e_1 + \delta_2 e_2) = \lambda_{\min>0}\left([\sqrt{\kappa} e_2, \sqrt{\kappa} e_1, \sqrt{\sigma}\mathcal{I}]^t [\sqrt{\kappa} e_2, \sqrt{\kappa} e_1, \sqrt{\sigma}\mathcal{I}]\right).$$

Let us denote by x the vector $\sqrt{\kappa} e_2$ and by X the matrix $[\sqrt{\kappa} e_1, \sqrt{\sigma}\mathcal{I}]$. Then, we have that

$$[\sqrt{\kappa} e_2, \sqrt{\kappa} e_1, \sqrt{\sigma}\mathcal{I}]^t [\sqrt{\kappa} e_2, \sqrt{\kappa} e_1, \sqrt{\sigma}\mathcal{I}] = \begin{bmatrix} x^t x & x^t X \\ X^t x & X^t X \end{bmatrix}$$

and using Theorem A.2 again, we obtain

$$\lambda_{\min>0}(\sigma L + \kappa e_1 e_1^t + \kappa e_2 e_2^t) \geq \lambda_{\min>0}(\sigma L + \kappa e_1 e_1^t) - \frac{\deg_2}{(\kappa - \lambda_{\min>0}(\sigma L + \kappa e_1 e_1^t))}.$$

Since $\lambda_{\min>0}(\sigma L + \kappa e_1 e_1^t) \leq \lambda_{\min>0}(\sigma L)$, we thus obtain

$$\lambda_{\min>0}(\sigma L + \kappa e_1 e_1^t + \kappa e_2 e_2^t) \geq \lambda_{\min>0}(\sigma L + \kappa e_1 e_1^t) - \frac{\deg_2}{(\kappa - \sigma \lambda_{\min>0}(L))}.$$

We can repeat the same argument r times and obtain

$$(2.5) \quad \lambda_{\min>0}(\sigma L + \kappa P) \geq \sigma \lambda_{\min>0}(L) - \frac{\sum_{i=1}^r \deg_i}{\kappa - \sigma \lambda_{\min>0}(L)}.$$

Finally, by Proposition 1.2, we know that the following constraint is sufficient for preserving controllability

$$(2.6) \quad \lambda_{\min>0}\left(\sigma L + \kappa \sum_{i=1}^r e_i e_i^t\right) \geq \frac{2 \|F_{\xi, \tilde{\xi}}\| \|Q\|}{\lambda_{\min}(QB + B^t Q^t)}.$$

By (2.5), it is sufficient to guarantee the controllability of our system to impose

$$\sigma \lambda_{\min>0}(L) - \frac{\sum_{i=1}^r \deg_i}{\kappa - \sigma \lambda_{\min>0}(L)} \geq \frac{2 \|F_{\xi, \tilde{\xi}}\| \|Q\|}{\lambda_{\min}(QB + B^t Q^t)}.$$

Then, combining (2.6) with (2.4) implies that

$$\kappa \geq \frac{\sum_{i=1}^r \deg_i}{\sigma \lambda_{\min>0}(L) - \frac{2 \|F_{\xi, \tilde{\xi}}\| \|Q\|}{\lambda_{\min}(QB + B^t Q^t)}} + \sigma \lambda_{\min>0}(L)$$

is a sufficient condition for controllability. \square

APPENDIX A. PERTURBATION THEORY OF EXTREME SINGULAR VALUES AFTER
APPENDING A COLUMN

A.1. Framework. Let d be an integer. Let $X \in \mathbb{R}^{d \times n}$ be a $d \times n$ -matrix and let $x \in \mathbb{R}^d$ be column vector. We denote by a subscript t the transpose of vectors and matrices. There exist at least two ways to study the singular values of the matrix (x, X) obtained by appending the column vector x to the matrix X :

(A1) Consider the matrix

$$(A.7) \quad A = \begin{bmatrix} x^t \\ X^t \end{bmatrix} \begin{bmatrix} x & X \end{bmatrix} = \begin{bmatrix} x^t x & x^t X \\ X^t x & X^t X \end{bmatrix};$$

(A2) Consider the matrix

$$\tilde{A} = \begin{bmatrix} x & X \end{bmatrix} \begin{bmatrix} x^t \\ X^t \end{bmatrix} = XX^t + xx^t.$$

On one hand, one may study in **(A1)** the eigenvalues of the $(n+1) \times (n+1)$ hermitian matrix A , i.e. the matrix $X^t X$ augmented with an arrow matrix.

On the other hand, one will deal in **(A2)** with the eigenvalues of the $d \times d$ hermitian matrix \tilde{A} , which may be seen as a rank-one perturbation of XX^t . The matrices A and \tilde{A} have the same non-zeros eigenvalues, and in particular $\lambda_{\max}(A) = \lambda_{\max}(\tilde{A})$. Moreover, the singular values of the matrix (x, X) are the square-root of the eigenvalues of the matrix A .

Equivalently, the problem of a rank-one perturbation can be rephrased as the one of controlling the perturbation of the singular values of a matrix after appending a column.

A.2. A theorem of Li and Li. In this paper, we use a slightly more general framework than **(A1)**, that is the case of a matrix

$$(A.8) \quad A = \begin{bmatrix} c & a^t \\ a & M \end{bmatrix},$$

where $a \in \mathbb{R}^d$, $c \in \mathbb{R}$ and $M \in \mathbb{R}^{d \times d}$ is a symmetric matrix.

The following theorem provides sharp upper bounds for $\lambda_{\max}(A)$, and lower bounds on $\lambda_{\min}(A)$, depending on various information on the sub-matrix M of A . As discussed above, this problem has close relationships with our problem of appending a column to a given rectangular matrix, because $\lambda_1(\tilde{A}) = \lambda_1(A)$.

Theorem A.1 (Li-Li's inequality and a lower bound). *Let d be a positive integer and let $M \in \mathbb{C}^{d \times d}$ be an Hermitian matrix, whose eigenvalues are $\lambda_1 \geq \dots \geq \lambda_d$ with corresponding eigenvectors (V_1, \dots, V_d) . Set $c \in \mathbb{R}$, $a \in \mathbb{C}^d$. Let A be given by (A.8). Therefore:*

$$(A.9) \quad \frac{2\langle a, V_1 \rangle^2}{\eta_1 + \sqrt{\eta_1^2 + 4\langle a, V_1 \rangle^2}} \leq \lambda_1(A) - \max(c, \lambda_1) \leq \frac{2\|a\|^2}{\eta_1 + \sqrt{\eta_1^2 + 4\|a\|^2}},$$

with

$$\eta_1 = |c - \lambda_1|.$$

A.3. Perturbation of the smallest nonzero eigenvalue. The same technics used to prove Theorem A.1 also give lower bounds for the smallest nonzero eigenvalue, which are also direct consequences of Li-Li's inequality. For more details, we refer the reader to [2].

Theorem A.2. *Let d be a positive integer and let $M \in \mathbb{C}^{d \times d}$ be a positive semi-definite Hermitian matrix, whose eigenvalues are $\lambda_1 \geq \dots \geq \lambda_d$ with corresponding eigenvectors (V_1, \dots, V_d) . Set $c \in \mathbb{R}$, $a \in \mathbb{C}^d$. Let A be given by (A.8). Assume that M has rank $r \leq d$. Therefore:*

$$(A.10) \quad \lambda_{r+1}(A) \geq \min(c, \lambda_r) - \frac{2\|a\|^2}{\eta_r + \sqrt{\eta_r^2 + 4\|a\|^2}},$$

with

$$\eta_r = |c - \lambda_r|.$$

In particular, the following perturbation bounds of Weyl and Mathias hold:

Corollary A.3.

$$(A.11) \quad \lambda_{r+1}(A) \geq \min(c, \lambda_r) - \|a\|_2$$

$$(A.12) \quad \lambda_{r+1}(A) \geq \min(c, \lambda_r) - \frac{\|a\|_2^2}{|c - \lambda_r|}.$$

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